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WILLIAM J. DAVIS

Department of Mathematics

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William J. Davis
Case Institute of Technology

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by

William J. Davis
Mathematics Department
Case Institute of Technology
University Circle
Cleveland, Ohio 44106

1. Introduction

This paper is concerned with an expansion theory for solving certain linear initial value problems in partial differential equations. In particular, the equations considered are to have classical solutions corresponding to arbitrary polynomial initial data functions.

For this class of problems, there exists a formal solution operator related to the equation, which transforms arbitrary polynomials in the space variable into classical solutions of the equation. The solution reduces to the given polynomial at the initial point. The derivation of this operator is an extension beyond the constant coefficients case of the derivation of the operational solutions of the Heaviside or Mikusinski operational calculus [10]. The transformation of polynomials further provides an immediate classical interpretation of the effects of the formal solution operator.

Through the use of this operator, generalized Appell sets of polynomials are transformed into basic solution sets for the equation. Series expansions in terms of these solution sets are studied. A similar expansion theory has been obtained for solutions of the heat equation by Widder and Rosenbloom [12]. In the parabolic case, the methods of Fourier transforms and generalized functions have been used to find generalized solutions, see [9].

The series representation theory here provides an alternative method for such problems, does not give existence or uniqueness theorems, but does increase the number of problems whose solutions may be examined explicitly. It further removes the consideration of the type of the partial differential equation.

The representation theory is developed here for a single equation in one space variable. We further consider only the first of the initial data conditions given, since the theory for all initial data functions is simply a vector generalization of this case. Generalized solutions are made to correspond to given initial data functions by the use of the theory of bi-orthogonal expansions in Banach spaces [7].

2. Solutions having polynomial initial data.

In this section we develop a method for solving the linear homogeneous initial value problem

$$P(D_x, D_t)\, U(x,t) = 0; \quad U(x,0) = Q(x), \tag{2-1}$$

where $Q(x)$ is a polynomial. The differential operator $P(D_x, D_t)$ in (2-1) has the form

$$P(D_x, D_t) = \sum_{i=0}^{L} \sum_{j=0}^{M} a_{ij}(x,t)\, D_x^i\, D_t^j , \tag{2-2}$$

where $D_x = \frac{\partial}{\partial x}$ and $D_t = \frac{\partial}{\partial t}$. Throughout this work, L and M are to denote the orders specified in (2-2). In the Mikusinski operational calculus, if the coefficients, a_{ij}, are constants, there exists an operational solution to (2-1) to be interpreted within the field of convolution quotients [10]. A similar operational solution, $F(x,t,D_x)$, is developed here which has the formal property that $F(x,t,D_x)Q(x) = U(x,t)$, the solution to problem (2-1), for arbitrary functions $Q(x)$. In the case that $Q(x)$ is a polynomial, this transform is a classical solution to the problem.

<u>Theorem 2-1:</u> Suppose $F(x,t,\lambda)$ is a classical solution to the <u>associated operator problem</u>.

$$P(D_x+\lambda,D_t)\ F(x,t,\lambda) = 0; \quad F(x,0,\lambda) = 1. \tag{2-3}$$

Then, $F(x,t,D_x)$ is a formal solution operator which transforms arbitrary polynomials, $Q(x)$, into classical solutions of (2-1).

<u>Proof</u>: Since polynomials are entire functions, $F(x,t,D_x)Q(x)$ becomes a classical solution to (2-1) whenever the operator identities

$$P(D_x,D_t)\ F(x,t,D_x) = 0; \quad F(x,0,D_x) = 1 \tag{2-4}$$

are satisfied. We get

$$P(D_x,D_t)F(x,t,D_x) = \tag{2-5}$$

$$\sum_{i=0}^{L}\sum_{j=0}^{M} a_{ij}(x,t)\ D_t^j \sum_{k=0}^{i} \binom{i}{k}\ D_1^{i-k}\ F(x,t,D_x)D_x^k ,$$

where we used Leibniz's rule, and where D_1 denotes differentiation with respect to the first argument. If we now replace D_x on the right hand side by λ, we get

$$\sum_{i=0}^{L}\sum_{j=0}^{M} a_{ij}(x,t)\ D_t^j \sum_{k=0}^{i} \binom{i}{k} D_1^{i-k}\lambda^k F(x,t,\lambda) = \tag{2-6}$$

$$P(D_x+\lambda,D_t)F(x,t,\lambda)\ .$$

This vanishes by (2-3). For the initial conditions, we get

$$F(x,0,D_x)Q(x) = \hat{1}\cdot Q(x) = Q(x).$$

This completes the proof.

If the coefficients in $P(D_x,D_t)$ do not depend upon x, then the formal operator itself need not depend upon x, for in that case, the equation in (2-3) does not depend upon x. The associated operator problem then becomes

$$P(\lambda,D_t)F(t,\lambda) = 0, \quad F(0,\lambda) = 1. \tag{2-7}$$

If λ represents generalized differentiation, this is just the Mikusinski form.

In order to simplify the interpretation of the formal solution operator acting on polynomials we shall consider in the sequel, only formal solution operators having the series form;

$$F(x,t,D_x) = \sum_{n=0}^{\infty} F_n(x,t)\, D_x^n \,. \tag{2-8}$$

The following corollary to theorem 2-1 shows that this form is always attainable in the case that (2-1) has a classical solution for arbitrary polynomials, $Q(x)$.

Corollary 2-1: If there is a classical solution $U_s(x,t)$ to (2-1) for $Q(x) = \frac{x^s}{s!}$; $s = 0,1,\ldots,$ then the functions

$$F_n(x,t) = \sum_{k=0}^{n} \frac{(-x)^{n-k}}{(n-k)!}\, U_k(x,t) \,, \tag{2-9}$$

are the coefficients of D_x^n in (2-8).

Proof: Suppose $F(x,t,D_x)$ has the form (2-8). For initial data e^{ax}, the solution is formally just $F(x,t,a)e^{ax}$. The exponential shift rule may be written as

$$P(D_x+\lambda,D_t)e^{-\lambda x}U(x,t,\lambda) = e^{-\lambda x}P(D_x,D_t)U(x,t,\lambda). \tag{2-10}$$

Therefore, if $U(x,t,\lambda)$ satisfies the equation in (2-1), and if $U(x,0,\lambda) = e^{\lambda x}$, then

$$F(x,t,\lambda) = e^{-\lambda x}\, U(x,t,\lambda) \tag{2-11}$$

is the desired solution to the associated operator problem. Now, let $\{U_s(x,t)\}_{s=0}^{\infty}$ be the solutions of the hypothesis. Then, the function

$$U(x,t,\lambda) = \sum_{s=0}^{\infty} U_s(x,t)\lambda^s \tag{2-12}$$

satisfies the requirements of (2-10,11), so that

$$F(x,t,\lambda) = e^{-\lambda x} \sum_{s=0}^{\infty} U_s(x,t)\lambda^s$$

is the solution of (2-3). This gives the form (2-9) for $F_n(x,t)$.

It may occur, as in [12], that each $F_n(x,t)$ is a polynomial. In that case, equation (2-8) then will give us that the solutions to (2-1) are themselves polynomials. Let N denote the degree of $Q(x)$. In any case,

$$U(x,t) = \sum_{m=0}^{N} F_m(x,t)\, Q^{(m)}(x). \tag{2-13}$$

For each $F_m(x,t)$ a polynomial, we get immediate degree relations in x and t for $U(x,t)$. Let M_n denote the degree in x of $F_n(x,t)$, and let N_n denote its degree in t. Then the degree of $U(x,t)$ in x is

$$\delta_x(U(x,t)) = \max_{0\leq j\leq N} [M_j + (M-j)] , \tag{2-14}$$

and in t is

$$\delta_t(U(x,t)) = \max_{0\leq j\leq N} N_j . \tag{2-15}$$

It is also clear at this point that the restriction on $Q(x)$, i.e. that it be a polynomial, may be removed if we are careful about convergence of $F(x,t,D_x)Q(x)$. If $Q(x)$ were analytic, this transform would exist formally as

$$U(x,t) = \sum_{m=0}^{\infty} F_m(x,t)\, Q^{(m)}(x) , \tag{2-16}$$

and $U(x,t)$ will be a solution under proper convergence of the series on the right hand side. For solution expansions over $-\infty < x < \infty$, such questions must be considered, but we restrict our attention to a finite interval in this development.

3. Generalizations of initial data and dimension.

Up to now we have discussed the formal solution operator only in the case of a single initial data function, and with the restriction that x be a one dimensional real variable. Although we shall retain these restrictions in the sequel, we indicate, in this section, the generalizations to the cases of several initial conditions, several space variables, and to systems of equations.

In order to properly pose a Cauchy problem for the equation in (2-1), we need to specify the M initial conditions

$$D_t^k\, U(x,0) = \varphi_k(x)\ ;\quad k = 0,1,\ldots, M-1, \tag{3-1}$$

where M denotes the order in D_t of $P(D_x,D_t)$. The formal solution operator for the problem thus attained may be found as follows. Let $F_j(x,t,\lambda)$ be a solution to the j^{th} associated problem:

$$P(D_x+\lambda,D_t)F_j(x,t,\lambda) = 0;\quad D_t^kF_j(x,0,\lambda) = \delta_{jk}, \tag{3-2}$$

for $j,k = 0,1,\ldots,M-1$. Here, as in the previous section, we may ask that $F_j(x,t,D_x)$ be given a series form. In this case,

$$F_j(x,t,D_x) = \sum_{n=-j}^{\infty} F_{jn}(x,t)D_x^n\ . \tag{3-3}$$

Then, let $F(x,t,D_x)$ denote the M-vector $(F_0(x,t,D_x),\ldots,F_{M-1}(x,t,D_x))$. Let $\Phi(x)$ denote the M-vector $(\varphi_0(x),\ldots, \varphi_{M-1}(x))$. Then, just as above, the scalar product of these vectors,

$$U(x,t) = F(x,t,D_x)\ \Phi(x), \tag{3-4}$$

is the desired formal solution to the given Cauchy problem.

We have also considered only problems in the single space variable x. If we allow x to denote $(x_1,\ldots,x_n)$, and define $D_x^\alpha = D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}$, equation forms such as (2-1) remain unchanged, and the associated operator equation is still

$$P(D_x + \lambda,D_t)F(x,t,\lambda) = 0, \tag{3-5}$$

where $\lambda = (\lambda_1,\ldots,\lambda_n)$. Expansions of initial data functions will generalize immediately to n-space if the domain of definition of these functions is contained in some product set, $\prod_{i=1}^{n} I_i$. This is due to the fact that

$$L_2\left(\prod_{i=1}^{n} I_i\right) = \prod_{i=1}^{n} L_2(I_i) .$$

Now consider a system of equations

$$\sum_{k=0}^{k} P_{jk}(D_x, D_t) U_k(x,t) = 0;\ j = 0,1,\dots, J. \tag{3-6}$$

If we use precisely the same arguments as in theorem 2-1, we find that the associated operator equation for this situation is just

$$\sum_{k=0}^{k} P_{jk}(D_x + \lambda, D_t)\ F_k(x,t,\lambda) = 0;\ j = 0,1,\dots, J . \tag{3-7}$$

Once again this is just a vector generalization of the situation which is treated in this work.

4. The solution representation problem.

We have determined a method for finding classical solutions to (2-1), in the case that $Q(x)$ is a polynomial (or perhaps an analytic function). We now extend the class of admissible initial data for such problems to certain subsets of $L_2(I)$, where I denotes some closed interval of the x-axis. If we wish I to be $(-\infty, \infty)$, we must extend theorem (2-1) to hold in the case that $Q(x)$ is a basis element for $L_2(-\infty,\infty)$ such as, perhaps, any Hermite orthogonal function. For any finite interval, the class of all polynomials will suffice for the representation of initial data due to the Weierstrass approximation theorem.

In particular, the polynomial approximations used shall be elements of the linear span of a simple set of generalized Appell polynomials. Boas and Buck [2,3] have shown that polynomials of degree precisely equal to n are generated by a function of the form

$$G(x,a) = A(a)\ \psi(xg(a)) \ , \tag{4-1}$$

in the sense that

$$G(x,a) = \sum_{n=0}^{\infty} p_n(x)a^n. \tag{4-2}$$

Here we must have

$$A(a) = \sum_{n=0}^{\infty} A_n a^n \ , \quad A_0 \neq 0, \tag{4-3}$$

$$\psi(s) = \sum_{n=0}^{\infty} \gamma_n s^n \ , \qquad \gamma_k \neq 0,\ k = 0,1,\ldots, \tag{4-4}$$

and

$$g(a) = \sum_{n=1}^{\infty} g_n a^n \ . \tag{4-5}$$

The above mentioned Hermite orthogonal functions have such a generator, except that in that case, condition (4-5) is violated.

Boas and Buck have studied representations in series form of analytic functions in terms of such sets. In order to attain expansions of $L_2(I)$ functions in terms of these polynomials, we rely upon the theory of biorthogonal expansions in Banach spaces. In particular, we use the presentation of this theory in [7].

In this way, if the set $\{p_n(x)\}$ is minimal (cf. § 5), we obtain series representations for initial data functions in the form

$$f(x) \sim \sum_{n=0}^{\infty} c_n p_n(x) \ . \tag{4-6}$$

We call the corresponding series

$$V(x,t) \sim \sum_{n=0}^{\infty} c_n p_n(x,t) \tag{4-7}$$

a <u>generalized solution</u> to (2-1), for $Q(x) = f(x)$. In (4-7), $p_n(x,t) = F(x,t,D_x)p_n(x)$, the solution to (2-1) for $Q(x) = p_n(x)$. A special solution expansion theory for the heat equation is handled in a similar manner by Widder and Rosenbloom [12]. In that case, the formal solution operator is $e^{tD_x^2}$, and the basic solutions are polynomials in x and t which are closely related to the Hermite polynomials.

The advantage in using generalized Appell polynomials in the expansion theory lies in the fact that the corresponding solution sets, $\{p_n(x,t)\}$, are generated by

$$H(x,t,a) = A(a)F(x,t,D_x)G(x,a). \tag{4-8}$$

Thus, if $c_n = a^n$ in (4-6) and (4-7) the convergence of the generalized solution is identical with that of the series for $H(x,t,a)$. It is precisely this correspondence that is basic to the subsequent convergence theory.

Let $R(x,t)$ be the radius of convergence of the series $\Sigma a^n p_n(x,t)$ at the point (x,t). By the radical test,

$$R(x,t) = \left[\overline{\lim_{n\to\infty}} \; |p_n(x,t)|^{1/n}\right]^{-1}. \tag{4-9}$$

Furthermore, the series in (4-7) converges at (x,t) if

$$\overline{\lim_{n\to\infty}} \; |c_n p_n(x,t)|^{1/n} < 1, \tag{4-10}$$

and diverges if

$$\overline{\lim_{n\to\infty}} \; |c_n p_n(x,t)|^{1/n} > 1. \tag{4-11}$$

Therefore, we may state the basic convergence criterion in this lemma.

<u>Lemma 4-1:</u> The series in (4-7) converges at (x,t) if

$$C < R(x,t), \tag{4-12}$$

and diverges there if

$$C > R(x,t), \tag{4-13}$$

where C is defined as

$$C = \overline{\lim_{n\to\infty}} \; |c_n|^{1/n} . \tag{4-14}$$

The case $C = R(x,t)$ is indeterminate by this test.

We call a sequence $\{C_n\}$ a <u>C-sequence</u> if and only if (4-14) holds. The constant C will be called <u>the growth</u> of $\{C_n\}$.

5. Expansions of initial data.

Here we determine the coefficients c_n in (4-6) corresponding to arbitrary $f(x)\varepsilon L_2(I)$. For these expansions, we use certain restricted sets of Generalized Appell polynomials. We rely on the theory of biorthogonal expansions in Banach space, in particular as presented by Frink [7]. For completeness, this work is summarized in the following paragraph, in the case that the Banach space in question is $L_2(I)$.

A sequence of polynomials $\{p_n(x)\}$ is said to be <u>minimal</u> in $L_2(I)$ if $p_n(x) \not\in \overline{\text{sp}\,\{p_m(x)\}_{m\neq n}}$. If $\{p_n(x)\}$ is minimal, and if $f(x)\varepsilon L_2(I)$, then there is a unique constant ξ_n such that $f(x) - \xi_n p_n(x)$ is in $\overline{\text{sp}\{p_m(x)\}_{m\neq n}}$. These constants determine continuous linear functionals on $L_2(I)$ by the relation $\xi_n = L_n(f)$. Therefore, since L_2 is a Hilbert space, there exists a set $\{w_n(x)\}\subset L_2(I)$ such that $L_n(f) = (w_n,f)$, and such that $(w_n,p_m) = \delta_{nm}$, the Kronecker delta. A system such as $\{p_n(x);\, w_m(x)\}$ is called biorthonormal, and is said to be complete if either $P = \{p_n(x)\}$ is complete or if $W = \{w_n(x)\}$ is complete. If $\{P;W\} = \{p_n(x);w_m(x)\}$ is complete, then both P and W are complete. The existence of the set W biorthonormal to P is sufficient for the minimality of P. Let $f(x) = \underset{n\to\infty}{\text{l.i.m.}} \sum_{k=0}^{n} c_{nk}p_k(x)$. Then, we get $\lim_{n\to\infty} c_{nk} = (w_k(x),f(x))$, a fact we use in determining the set $\{c_k\}$.

Since the minimality of P is necessary and sufficient for the existence of the set W, we shall consider generalized Appell sets which are minimal in $L_2(I)$, where now I is either the domain of interest for the problem, or some containing interval. In certain cases, the nonminimality of P may be determined by an examination of the generator, $A(a)\ \psi(xg(a))$ [3]. This lemma further indicates that minimality is also closely allied to the domain of $L_2(I)$.

Lemma 5-1: Let the complete set $\{p_n(x)\}$ be minimal in $L_2(I)$. Then P is not minimal in $L_2(J)$ if $J \supset I$ and $J - I$ has positive measure, or in $L_2(J)$ if $I \supset J$ and $I - J$ has positive measure.

Proof: Suppose first that $I \supset J$. Let $f(x) \varepsilon L_2(J)$, and $g(x) \varepsilon L_2(I)$ such that, in $I - J$, $g(x) \neq 0$ on a set of positive measure. Define $\varphi(x)$ and $\psi(x)$ thusly:

$$\varphi(x) = \begin{cases} f(x) & ;\ x \varepsilon J \\ 0 & ;\ x \varepsilon I-J, \end{cases}$$

and

$$\psi(x) = \begin{cases} f(x) & ;\ x \varepsilon J \\ g(x) & ;\ x \varepsilon I-J. \end{cases}$$

Both $\varphi(x)$ and $\psi(x)$ are in $L_2(I)$, and determine unique sequences $\{\varphi_n\}$ and $\{\psi_n\}$ which are not equal. Consider the sequence $\{\alpha_n\} = \{\varphi_n - \psi_n\}$. The function determined by this sequence is zero for $x \varepsilon J$, so in $L_2(J)$, if $\alpha_k \neq 0$, $p_k(x) = -\frac{1}{\alpha_k} \sum_{n \neq k} \alpha_n p_n(x)$, which contradicts the minimality of P in $L_2(J)$. The argument for $J \supset I$ follows immediately from this.

We now turn to the problem of determining the set $\{w_n(x)\}$ biorthogonal to a minimal generalized Appell set $\{p_n(x)\}$. Suppose that $G(x,a)$ is analytic in a for all $x \varepsilon I$ for $|a| \leq \max(|\alpha|, |\beta|)$. Here α and β denote the endpoints of I. Consider the Fredholm integral equation of the first kind,

$$f(x) = \int_I G(x,y)F(y)dy. \tag{5-1}$$

Upon substitution of the series form of $G(x,y)$, this becomes

$$f(x) = \sum_{n=0}^{\infty} p_n(x) \int_I y^n F(y)dy \;. \tag{5-2}$$

If there is a function $F(y)$ such that (5-1) is satisfied and the integral on the right is almost uniformly continuous, then there is an expansion for $f(x)$ in terms of $\{p_n(x)\}$ with $c_n = \int y^n F(y)dy$.

Let $K_1(x,y) = \int_I G(x,z)G(y,z)dz$ and let $K_2(x,y) = \int_I G(z,x)G(z,y)dz$. Then there exist sets of eigenvalues and eigenfunctions, $\{\lambda_n\}, \{\varphi_n(x)\}$ and $\{\psi_n(x)\}$ such that

$$\varphi_n(x) = \lambda_n \int_I G(x,y)\ \psi_n(y)dy \tag{5-3}$$

$$\varphi_n(x) = \lambda_n^2 \int_I K_1(x,y)\ \varphi_n(y)dy \tag{5-4}$$

$$\psi_n(x) = \lambda_n \int_I G(y,x)\ \varphi_n(y)dy \tag{5-5}$$

$$\psi_n(x) = \lambda_n^2 \int_I K_2(x,y)\ \psi_n(y)dy, \tag{5-6}$$

since the kernels K_1 and K_2 are symmetric. By a theorem of Picard [4], a necessary and sufficient condition for the existence of a square integrable solution to (5-1) is that

$$\sum_{n=0}^{\infty} \lambda_n^2 \left(\int_I f(x)\varphi_n(x)dx \right)^2 \tag{5-7}$$

be convergent. Thus, we have the following lemma.

<u>Lemma 5-2</u>: If the series (5-7) is convergent, there exists $F(y) \in L_2(I)$ such that

$$(f, w_n) = (y^n, F(y)). \tag{5-8}$$

Let $\{p_n(x)\}$ be a simple minimal set of polynomials in $L_2(I)$. Let $\{\psi_n(x)\}$ be a complete orthonormal set of polynomials for $L_2(I)$. For any $f(x) \in L_2(I)$ we have

$$f(x) = \sum_{n=0}^{\infty} (f,\psi_n)\psi_n(x) \tag{5-9}$$

in the mean on I. Also, by the continuity imposed upon $G(x,y)$ for $(x,y) \varepsilon I \times I$, we have

$$G(x,y) = \sum_{n=0}^{\infty} X_k(x)\ \psi_k(y), \quad \text{and} \tag{5-10}$$

$$G(x,y) = \sum_{n=0}^{\infty} \psi_k(x)\ Y_k(y)\ . \tag{5-11}$$

From (5-8) and (5-11) we get, formally,

$$\sum_{k=0}^{\infty} \psi_k(x)(Y_k,F) = \sum_{n=0}^{\infty} p_n(x)(y^n,F(y)). \tag{5-12}$$

Since $\{\psi_k(x)\}$ is a complete orthonormal set of polynomials we have

$$\psi_k(x) = \sum_{m=0}^{k} \alpha_{km} p_m(x). \tag{5-13}$$

Therefore, from (5-12) and (5-13) we get formally

$$w_n(x) = \sum_{k=n}^{\infty} \alpha_{kn}\psi_k(x) = \sum_{k=n}^{\infty} \sum_{\ell=0}^{k} \alpha_{kn}\alpha_{k\ell}\, p_\ell(x). \tag{5-14}$$

The convergence in the mean of the first series in this last expression is guaranteed by the minimality of $\{p_n(x)\}$. This yields the theorem.

Theorem 5.1: If $\{p_n(x)\} \subset L_2(I)$ is a minimal simple set of polynomials, then the sequence $\{w_n(x)\}$ is a subset of $L_2(I)$, and the system $\{p_n;w_m\}$ is complete and biorthonormal in $L_2(I)$.

6. Convergence of generalized solutions.

We now turn to the problem of convergence of the generalized solutions, (4-9). The basis for this theory is lemma 4-1, which is used to determine point sets in the plane in which the series converges. Let $R(x,t)$ be as defined in (4-9), and let

$$Q(x,t) = \left[\lim_{n \to \infty} |q_n(x,t)|^{1/n} \right]^{-1} , \qquad (6\text{-}1)$$

where $q_n(x,t) = D_x^L D_t^M p_n(x,t)$. Throughout this section, (L,M) denotes the order of the operator, $P(D_x, D_t)$, of the equation in (2-1). Let $U(x,t)$ denote the series in (4-7), and define

$$W(x,t) \sim \sum_{n=0}^{\infty} c_n q_n(x,t) \; . \qquad (6\text{-}2)$$

It is clear that $Q(x,t)$ is the radius of convergence of $\Sigma b^n q_n(x,t)$. Consider $R(x,t)$ and $Q(x,t)$ as functions of x and t. If we admit the possible value $+\infty$ into the ranges of these functions, then each of them is a well-defined, non-negative function for all (x,t) in the plane.

Fix C such that, $0 \leq C \leq \infty$. Let us now define the point sets $U_C(x,t)$, $V_C(x,t)$ and $C_C(x,t)$ by the relations

$$U_C(x,t) = \{(x,t) \mid C < P(x,t)\} , \qquad (6\text{-}3)$$

$$V_C(x,t) = \{(x,t) \mid C \leq P(x,t)\} , \text{ and} \qquad (6\text{-}4)$$

$$C_C(x,t) = \{(x,t) \mid C < Q(x,t)\} . \qquad (6\text{-}5)$$

We now derive the containment and convergence relations which justify the above definitions of generalized solutions. It is clear from the defining relations that $U_C(x,t) \subseteq V_C(x,t)$. Also, if $C > D$, then $U_C(x,t) \subseteq U_D(x,t)$, $V_C(x,t) \subseteq V_D(x,t)$ and $C_C(x,t) \subseteq C_D(x,t)$.

<u>Lemma 6-1:</u> Let $\{c_n\}$ be a C-sequence. Then, the series for $U(x,t)$ converges inside $U_C(x,t)$, and diverges outside $V_C(x,t)$. The series, (6-2), for $W(x,t)$ converges inside $C_C(x,t)$.

<u>Proof:</u> By lemma 4-1, $U(x,t)$ converges for $C < R(x,t)$, ie for $(x,t) \in U_C(x,t)$. Also, $U(x,t)$ diverges for $C > R(x,t)$, for $(x,t) \notin V_C(x,t)$. Finally, $W(x,t)$ converges for $C < Q(x,t)$, ie. for $(x,t) \in C_C(x,t)$.

This next lemma provides an alternative description of the sets $U_C(x,t)$ and $C_C(x,t)$. First, let $\pi(x,t) = \{(x,t) | p_n(x,t) \neq 0$ for only finitely many integers $n\}$. On $\pi(x,t)$ it is clear that $R(x,t)$ and $Q(x,t)$ are both equal to $+\infty$.

<u>Lemma 6-2:</u> Let $\Gamma = \{\{c_n\}\}$ be the collection of all C-sequences for some fixed C. For $\{c_n\} \in \Gamma$, let $U(\{c_n\}, x,t)$ be the set of all points at which $U(x,t)$ converges. Let $C(\{c_n\}, x,t)$ be the set of points at which $W(x,t)$ converges. Then,

$$\cap\, U\,(\{c_n\},\ x,t) = U_C(x,t)\ ,\ \text{and} \tag{6-6}$$

$$\cap\, C\,(\{c_n\},\ x,\ t) = C_C(x,t)\ , \tag{6-7}$$

where the intersections are taken over all elements of Γ.

__Proof:__ First consider $\cap U(\{c_n\}, x,t)$. It is clear that $\pi(x,t) \subseteq U_C(x,t) \subseteq \cap U(\{c_n\}, x,t)$, since any C-sequence defines a convergent generalized solution inside $U_C(x,t)$. Let (x,t) be an element of $\cap U(\{c_n\}, x,t) - U_C(x,t)$. At this point, $R(x,t) = C$, since each $U(\{c_n\}, x,t) \subseteq V_C(x,t)$ by the previous lemma. Let $c_n = [p_n(x,t)]^{-1}$ if $p_n(x,t) \neq 0$, and let $c_n = 0$ if $p_n(x,t) = 0$. Then, there is a subsequence of $\{c_n\}$ such that $\lim |c_n|^{1/n} = R(x,t)$, by the definition of $R(x,t)$. Therefore, set $c_n = 0$ also if c_n is not an element of such a subsequence. The sequence thus constructed is a C-sequence. Its associated generalized solution diverges since the series in (4-7) becomes an infinite sum of 1's. Therefore, the set $\cap U(\{c_n\},x,t) - U_C(x,t) = \emptyset$. The proof of the second part is precisely the same.

We now establish the key theorem in the correspondence of $U(x,t)$ to classical and Sobolev generalized continuous solutions [11].

__Theorem 6-1:__ The series (4-7) defining $U(x,t)$ converges to a continuous generalized solution of the equation in (2-1) on compact subsets of $U_C(x,t)$. This series converges to a classical solution of the equation on compact subsets of $C_C(x,t)$ which contain the set $I \times \{0\}$.

__Proof:__ Let X be a compact subset of $U_C(x,t)$. Let

$$S_N(x,t) = \sum_{n=0}^{N} c_n p_n(x,t) . \tag{6-8}$$

$S_N(x,t)$ is a classical solution to (2-1) with $Q(x) = \sum_{n=0}^{N} c_n p_n(x)$.

By lemma 6-1, $\{S_N(x,t)\}$ converges pointwise in X, and so the sequence converges uniformly to a continuous function on X. Such a limit is a Sobolev generalized solution.

Let Y be a compact subset of $C_C(x,t)$, such that $Y \supseteq I \times \{0\}$. Then, by the argument above, $W(x,t)$ converges uniformly in Y to a continuous function. With suitable choices of integration functions, $p_n(x,t) = \int \cdots \int q_n(y,s)\, dy^L\, ds^M$. Integrating the series for $W(x,t)$ term by term (L,M) times in this way, we get $U(x,t)$. By the uniform convergence of the series for $W(x,t)$, the limit of $\{S_N(x,t)\}$ is $U(x,t)$, and is L times continuously differentiable in x and M times continuously differentiable in t. Since $I \times \{0\} \subset Y$, $\lim_{t \to 0} U(x,t)$ exists, is L times continuously differentiable and represents the initial data function for the problem.

This theorem also gives us a further containment relation between sets. A single point is a compact set in the plane, so the series for $W(x,t)$ converges whenever $\{S_N(x,t)\}$ converges. Thus, we have the relations

$$C_C(x,t) \subseteq U_C(x,t) \quad , \quad Q(x,t) \leq P(x,t). \qquad (6\text{-}9)$$

In order to develop a correspondence with the theory of distributions, we need some characterization of measurable sets on which the series (4-7) converges in the mean. Let $R(\{c_n\},x,t)$

be such a set for a given sequence $\{c_n\}$. That is, suppose the series in (4-7) converges in the mean on $R(\{c_n\},x,t)$. Such sets exist if $U_C(x,t)$ contains a measurable compact subset. This is due to the fact that, on such sets, $U(x,t)$ is uniformly continuous, and therefore square integrable. We also obtain the following lemma.

<u>Lemma 6-3</u>: Let Y be a connected subset of $R(\{c_n\},x,t)$ with positive measure. Then, $Y \subseteq \overline{V_C(x,t)}$, where the bar denotes Euclidean closure.

<u>Proof</u>: The series for $U(x,t)$ converges in the mean on R, and therefore in measure in Y. Let Y' be the points of Y at which the series (4-7) converges. Then, $Y - Y'$ has zero measure, and Y is connected, so Y' is dense in Y. Also, $Y' \subseteq V_C(x,t)$ by lemma 6-1. It follows that

$$Y \subseteq \overline{Y'} \subseteq \overline{V_C(x,t)}.$$

By definition, the partial sums of (6-8) converge in the mean on $R(\{c_n\}, x,t)$, to a Sobolev, $L_2(R)$, generalized solution of the equation of (2-1). Now we develop an immediate connection with the theory of distributions as applied to partial differential equations [8].

<u>Lemma 6-4</u>: Let $\varphi(x,t)$ be an arbitrary element of the testing space $C_0^\infty(R(\{c_n\}, x,t))$. Let $U(x,t)$ be the $L_2|R|$ limit of the partial sums in (6-8). Then the distribution of function type defined by

$$U(\varphi) = \int_R U(x,t)\, \varphi(x,t)\, dx\, dt$$

is a solution in the distribution sense to the equation in (2-1).

Proof: Let ${}^{T}P(D_x, D_t)$ denote the formal adjoint of $P(D_x, D_t)$.

Then,

$$P(D_x, D_t)\ U(\varphi) = \int_R U(x,t)\,{}^{T}P(D_x, D_t)\ \varphi(x,t)dx\ dt$$

$$= \underset{N\to\infty}{\text{l.i.m.}} \int_R S_N(x,t)({}^{T}P\varphi)dx\ dt = \underset{N\to\infty}{\text{l.i.m.}} \int_R [P(D_x, D_t)S_N(x,t)]\ \varphi(x,t)dx\ dt. \tag{6-10}$$

By the Lebesgue dominated convergence theorem, since the integral of this limit exists, we get immediately that $P(D_x, D_t)\ U(\varphi) = 0$.

We may classify initial data functions according to the growths of their pseudo-Fourier coefficients, c_n. That is, let $f(x) \in L_2(I)$. Then we say that $f(x)$ has growth C if $\{c_n\}$ is a C-sequence. This definition, together with the convergence theory developed up to this point, prompts the following definition.

Definition 6-1: A function, $f(x) \in L_2(I)$, is a generalized initial data function if its growth is finite. It is generalized L_2 initial data if there exists a set $R(\{c_n\}, x,t)$ containing $I \times \{0\}$. The function $f(x)$ is generalized continuous initial data if $U_C(x,t) \supseteq I \times\{0\}$, and is classical initial data if $C_C(x,t) \supseteq I \times \{0\}$.

The following lemma is concerned with the existence or non-existence of the convergence sets $U_C(x,t)$. As becomes apparent in the proof, the lemma holds as well for the sets $V_C(x,t)$, $C_C(x,t)$ and $R(\{c_n\}, x,t)$.

Lemma 6-5: There exist void convergence sets $U_C(x,t)$ if and only if $R(x,t)$ is bounded above. If $R(x,t)$ is bounded below by M, then $U_C(x,t)$ is the whole plane for all $C \leq M$.

Proof: If $U_C(x,t) = \emptyset$ for some C, then there is no point (x,t) such that $C < R(x,t)$. Such a C serves as an upper bound for $R(x,t)$. If $R(x,t) \leq N$, then $U_C(x,t) = \emptyset$ for all $C > N$. If $R(x,t) \geq M > C$, then the defining relation is satisfied at all points of the plane.

In the preceding work of this section, we have considered the sequence $\{c_n\}$ fixed, and have then determined the corresponding convergence sets. In general, however, we wish to determine convergence of generalized solution series for the problem

$$P(D_x, D_t)\, U(x,t) = 0 \quad \text{for} \quad (x,t) \in I \times E, \tag{6-11}$$

$$U(x,0) = f(x) \in L_2(I), \tag{6-12}$$

where I and E are x and t intervals, respectively, and where $0 \in E$. We answer this convergence question in the next theorem. Let

$$R_R = \inf_{I \times E} R(x,t), \quad \text{and } Q_R = \inf_{I \times E} Q(x,t). \tag{6-13}$$

Theorem 6-2: Let $f(x) \in L_2(I)$ have growth C. If $C < R_R$, then the series for $U(x,t)$ converges to a generalized continuous solution to problem (6-11,12). If $C < Q_R$, the series for $U(x,t)$ converges to a classical solution of (6-11,12). The series for $U(x,t)$ defines an $L_2(I \times E)$ generalized solution to the problem only if $C \leq \operatorname*{ess\,inf}_{I \times E} R(x,t)$.

Proof: Under the hypotheses, $C < R_R$ implies that $U_C(x,t) \supseteq I \times E$, and $C < Q_R$ implies that $C_C(x,t) \supseteq I \times E$. The rectangle $I \times E$ is compact, so theorem 6-1 gives the first two conclusions. If $C >$ ess. inf $R(x,t)$, there is a set of positive measure in $I \times E$ on which $C > R(x,t)$. On this subset, the series for $U(x,t)$ must diverge, so that $U(x,t)$ cannot converge in the mean on $I \times E$.

7. The analyticity theorem.

The theorem of this section provides a method for determining the convergence radius, $R(x,t)$, used in the definition of the convergence sets $U_C(x,t)$ and $V_C(x,t)$. The method is based on the special form of the Boas-Buck generator for the basic polynomial sets. In case $G(x,a) = e^{ax}$ the determination of $R(x,t)$ is immediate. Let $r(x,t)$ be the radius of convergence of $F(x,t,\lambda)$. Then,

$$F(x,t,D_x)\ G(x,a) = e^{ax}\ F(x,t,a),$$

so that $r(x,t) = R(x,t)$. A direct generalization of the analyticity theorem will determine $Q(x,t)$ and the set $C_C(x,t)$. We state and prove this theorem in several parts to ease the consideration of separate cases.

Suppose, first, that the generator $G(x,a)$ is an entire function of a at the point x. Then, $\{p_n(x)\}$ satisfies

$$\overline{\lim_{n \to \infty}}\ |\ p_n(x)\ |^{1/n} = 0. \tag{7-1}$$

Consider

$$F(x\ ,t,D_x)\ G(x,a) = A(a) \sum_{n=0}^{\infty} F_n(x,t)g^n(a)\ \psi^{(n)}(xg(a)),$$

at the point (x,t). Since $G(x,a)$ is entire, $A(a)$, $g(a)$ and $\psi(s)$ are all entire. We now determine conditions on the formal solution operator, $F(x,t,D_x)$, which force the transformed generator, $H(x,t,a)$, to be an entire function of a. If $\Phi(t)$ is an entire function of t with growth (ρ,τ), then $\Phi^k(t)$ is again entire and of growth (ρ,τ), [1]. Since we are interested only in growth arguments, we consider the sum

$$A(a)\ \psi(xg(a))\left[\sum_{n=0}^{\infty} F_n(x,t)\ g^n(a)\right], \tag{7-3}$$

instead of the more complicated form in (7-2). The function defined by (7-3) is, therefore, entire in a whenever $F(x,t,g(a))$ is entire in $g(a)$. This proves the first part of the theorem.

<u>Theorem 7-1.1</u>: If $F(x,t,\lambda)$ and $G(x,a)$ are entire functions of λ and a at (x,t) and x repsectively, then $H(x,t,a)$ is an entire function of a at (x,t), and $R(x,t) = +\infty$.

If $F(x,t,\lambda)$ is a regular analytic function of λ at (x,t), let $r(x,t)$ denote its radius of convergence there. Then (7-3) represents an analytic function of a if $|g(a)| < r(x,t)$. Let

$$A(x,t) = \inf\{|a| \ \big|\ |g(a)| = r(x,t)\}. \tag{7-4}$$

The function $A(x,t)$ is clearly greater than zero if $r(x,t)$ is, since $g(0) = 0$ and $g(a) \not\equiv 0$. This gives us the second portion of this theorem.

<u>Theorem 7-1.2</u>: Suppose that, at (x,t), $G(x,a)$ is an entire function of a and $F(x,t,\lambda)$ is an analytic function of λ for $|\lambda| < r(x,t)$. Then $H(x,t,a)$ is analytic and $R(x,t) = A(x,t)$, where $A(x,t)$ is defined by (7-4).

Now suppose that $G(x,a)$ is just a regular function of a at x and has the radius of convergence $\rho(x)$. Suppose first that g is entire. Then, if σ is the radius of convergence of $\psi(s)$,

$$\rho(x) = \inf \{ |a| \; \Big| \; |g(a)| = \frac{\sigma}{|x|} \} . \tag{7-5}$$

Here we note that $\rho(x)$ must be smaller than the radius of convergence of $g(a)$ for all non-zero x, and is equal to this radius at $x = 0$.

If $F(x,t, \lambda)$ is an entire function of λ at (x,t), the series on the right in (7-2) will converge absolutely and uniformly if

$$\overline{\lim_{n \to \infty}} \, | \, g^n(a) \;\; \psi^{(n)}(xg(a)) \, |^{1/n} < \infty . \tag{7-6}$$

Since $g(a)$ is finite for $|a| < \rho(x)$, this holds if

$$\overline{\lim_{n \to \infty}} \, |(\psi^{(n)}(xg(a)) \, |^{1/n} < \infty. \tag{7-7}$$

The function $\psi(s)$ is analytic, so $\psi^{(n)}(s)$ has the same radius of convergence. However, at a fixed point s, $\psi^{(n)}(s)$ may grow as rapidly as $n!$ as n increases. We must, therefore, consider this case in a different way. We have

$$F(x,t,D_x)\ G(x,a) = A(a) \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} (m)_n\ x^{-n}\ F_n(x,t) \right\} \gamma_m\ [xg(a)]^m\ . \tag{7-8}$$

Let $b(x,t)$ denote the radius of convergence of this series as a power series in $g(a)$. Using the radical test, we get

$$b^{-1}(x,t) = \frac{1}{\rho(x)} \overline{\lim_{m\to\infty}} \left| \sum_{n=0}^{m} (m)_n\ x^{-n} F_n(x,t) \right|^{1/m}. \tag{7-9}$$

The $\overline{\lim}$ on the right hand side of (7-9) may be thought of as the reciprocal of the radius of convergence of the series

$$\varphi(x,t,y) = \sum_{m=0}^{\infty} y_m \sum_{n=0}^{m} (m)_n\ x^{-n} F_n(x,t)$$

in the variable y. Changing orders of summation, we get

$$\varphi(x,t,y) = \sum_{n=0}^{\infty} n!\ F_n(x,t)\ (y/x)^n\ \frac{1}{(1-y)^{n+1}}\ .$$

As a series in $y[x-xy]^{-1}$, this series has a radius of convergence given by $d(x,t)$, where

$$d^{-1}(x,t) = \overline{\lim_{n\to\infty}}\ [(n!)^{1/n}\ (F_n(x,t)|^{1/n}]\ . \tag{7-10}$$

If $r(x,t)$ is finite, $d(x,t) = 0$ since $(n!)^{1/n} \xrightarrow{\ \ } \infty$ with n. If $r(x,t) = +\infty$, and if $d(x,t) > 0$, the series for $\varphi(x,t,y)$ converges for $|y| < 1$, and $|y| < |x||\ 1 - y|d(x,t)$. That is, the series converges for $y < \min\ (1, \frac{d}{1+d}) =$ $d(x,t)\ [1 + d(x,t)]^{-1}$. Therefore, (7-9) gives

$$b(x,t) = \rho(x)\left[\frac{d(x,t)}{1+d(x,t)}\right] \qquad (7\text{-}11)$$

This gives the third part of the theorem.

Theorem 7-1.3: If $\psi(s)$ has radius of convergence σ, then $H(x,t,a)$ has radius of convergence in a at (x,t) given by

$$R(x,t) = \inf\{|a| \; ; \; |g(a)| = b(x,t)\}. \qquad (7\text{-}12)$$

From equation (7-10), we obtain the following corollary.

Corollary 7-1: If $F(x,t,\lambda)$ and $\psi(s)$ are both analytic functions with finite convergence radii, $H(x,t,a)$ diverges for all non-zero a, and $R(x,t) = 0$.

Another example in which $R(x,t) = 0$ is given by the formal operator e^{tD^2} connected with the heat equation, with the Boas-Buck generator $G(x,a) = [1-ax]^{-1}$. In this case,

$$H(x,t,a) = \sum_{k=0}^{\infty} \frac{(2k)!}{k!} t^k \frac{a^{2k}}{(1-ax)^{2k+1}}$$

which diverges for all non-zero choices of x,t and a. This is due to the fact that (7-10) becomes

$$d^{-1}(x,t) = \overline{\lim_{n\to\infty}} \left|\frac{(2n)!}{n!} t^n\right|^{1/n} = +\infty.$$

This completes the theorem except for the consideration of the effects of $A(a)$. Since $A(a)$ merely multiplies $F(x,t,D_x)\ \psi(xg(a))$, if $A(a)$ has a radius of convergence A, then in any case, the radius $R(x,t)$ will just be the minimum of A and the convergence radius determined in the theorem.

To determine $Q(x,t)$ we must examine the radius of convergence of $D_x^L D_t^M H(x,t,a)$. This is just

$$D_t^M\left[\sum_{m=0}^{L}\binom{L}{m}\left[D_x^{L-m}\ F(x,t,D_x)\right]\left[D_x^m G(x,a)\right]\right] .$$

Therefore, we may use the theorem to find $Q(x,t)$ by finding the radius of convergence for each function

$$\left[D_x^{L-m}\ D_t^M\ F(x,t,D_x)\right]\left[\ D_x^m\ G(x,a)\right] \quad ;\ m = 0,1,\ldots,L.$$

8. The well posed problem.

In this, for simplicity, assume that $P(D_x,D_t)$ is first order in t. Suppose the problem (6-11,12) is well posed in the rectangle $I \times E$. That is, suppose (6-11) has a unique solution corresponding to any classical initial data function, and that the solution depends continuously upon the initial data. The first lemma shows that $I\times E \subseteq V_C(x,t)$ with $C = \sup_{X\varepsilon I} P(x)$. The function $P(x)$ is the convergence radius of the generator $G(x,a)$, at the point x.

<u>Lemma 8-1</u>: Let (8-11,12) be well posed, and let $f(x)$ be classical initial data. Then the unique solution to the problem is the function $U(x,t)$ defined by (5-2).

<u>Proof</u>: The Weierstrass approximation and the analysis of section 5 guarantee that $\Sigma(f,w_n)p_n(x)$ will converge uniformly on I to $f(x)$. This series is also L-times continuously differentiable term by term on I. Let $S_N(x) = \sum_{n=0}^{N}(f,w_n)p_n(x)$. Then, since each $S_N(x,t)$ of (6-8) is a classical solution to (6-11) with $S_N(x,0) = S_N(x)$, the well posedness gives us the desired result. That is,

$S_N(x,t) \longrightarrow U(x,t)$, and that $U(x,t)$ is the classical solution to (6-11,12).

Using the above expansion theory to obtain solutions to (6-11,12), it is natural to ask when the partial sums converge to a known solution to the problem. That is, if $S_N(x,t) \longrightarrow U(x,t)$ for all classical initial data, and if $U(x,t)$ is a classical solution, what can we say about the problem? This is answered in the following theorem. Let $C^{L,M}(I \times E)$ denote the class of all functions, $H(x,t)$ which are L and M times continuously differentiable in the arguments x and t respectively. The norm which guarantees the completeness of this space is defined by

$$||h(x,t)||_{L,M} = \sum_{\substack{i \leq L \\ j \leq M}} \sup_{(x,t)\varepsilon I x E} \left[|D_x^i \, D_t^j h(x,t)| \right] . \tag{8-1}$$

Let $C^L(I)$ denote the corresponding one variable space with the norm

$$||f(x)||_L = \sum_{i \leq L} \sup_{X \varepsilon I} \left[|D_x^i f(x)| \right] . \tag{8-2}$$

<u>Theorem 8-1</u>: Let $f(x) \; \varepsilon \; C^L(I)$, so that $f(x)$ is classical initial data. Suppose that

$$\lim_{N \longrightarrow \infty} ||S_N(x,t) - U(x,t)||_{L,M} = 0, \tag{8-3}$$

whenever $U(x,t)$ is a classical solution to (6-11) with $U(x,0) = f(x)$. Then, problem (6-11,12) is well posed.

<u>Proof</u>: The uniqueness is obvious, for if $U(x,t)$ and $V(x,t)$ were two solutions with initial data $f(x)$, (8-3) would fail for one of them unless $||U(x,t) - V(x,t)||_{L,M} = 0$. This says $U(x,t) = V(x,t)$ in $C^{L,M}(IxE)$.

We now show the continuous dependence on initial data. Let $\{f_n(x)\} \subset C^L(I)$ such that

$$||f_N(x) - f(x)||_L \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty \tag{8-4}$$

As above, each $f_N(x)$ has an expansion in terms of the set $\{p_n(x)\}$. Let $S_{MN}(x,t)$ be defined by

$$S_{MN}(x,t) = \sum_{n=0}^{N} (f_M, w_n)\, p_n(x,t). \tag{8-5}$$

By (8-3), there is a sequence of solutions $\{U_M(x,t)\}$ such that $S_{MN} \longrightarrow U_M$ for each M. Let $\{\epsilon_n\}$ be a sequence of positive reals such that $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. Then, (8-3) may be restated as follows:

For each M, there exists $N(M)$ such that $N \geq N(M)$ implies that

$$||S_{MN}(x,t) - U_M(x,t||_{L,M} < \epsilon_M. \tag{8-6}$$

Also, since $f_M(x) \longrightarrow f(x)$ in $C^L(I)$, Frink's work guarantees that $(f_M, w_n) \longrightarrow (f, w_n)$ for each n as $M \longrightarrow \infty$. Therefore, for each N, there exists $M(N)$ such that $M \geq M(N)$ implies that

$$||S_{MN}(x,t) - S_{\infty N}(x,t)\,||_{L,M} < \epsilon_N. \tag{8-7}$$

Letting $\{(M,N)\}$ be partially ordered by

$$\{(M_1,N_1) \leq (M_2,N_2) \text{ if and only if } M_1 \leq M_2 \text{ and } N_1 \leq N_2\}, \tag{8-8}$$

the completeness of $C^{L,M}(I\mathrm{x}E)$ guarantees the convergence of $\{S_{MN}(x,t)\}$ to $U(x,t)$, due to a theorem of E.H. Moore [6]. Therefore, for each n there exists (M_n,N_n) such that

$(M,N) \geq (M_n,N_n)$ implies that

$$||U_M(x,t) - U(x,t)||_{L,M} \leq ||U_M(x,t) - S_{MN}(x,t)||_{L,M}$$

$$+||S_{MN}(x,t) - U(x,t)||_{L,M} < 2\varepsilon_n. \qquad (8\text{-}9)$$

Therefore, $U(x,t)$ depends continuously upon $f(x)$.

It is clear that these results apply also to the weaker forms of solutions. We need only ask, in the definition of well posedness, that solutions are Sobolev generalized solutions.

9. Solution bounds and stability.

Here we obtain bounds, valid inside convergence sets, for convergent generalized solutions. Let $I \times E \subseteq U_C(x,t)$. Let $f(x)$ be generalized continuous initial data for (6-12). Then, $(f,w_n) = c_n$ such that $\{c_n\}$ is a C sequence.

Recalling that $H(x,t,a)$ is analytic in a, we have that

$$p_n(x,t) = \frac{1}{n!} \frac{\partial^n}{\partial a^n} H(x,t,a) \mid a = 0 . \qquad (9\text{-}1)$$

Using the Cauchy integral form, we get the basic inequality which is valid for $\rho < R(x,t)$:

$$|p_n(x,t)| \leq \rho^{-n} M_H(x,t,\rho). \qquad (9\text{-}2)$$

The function $M_H(x,t,\rho)$ is given by

$$M_H(x,t,\rho) = \sup_{|Z|=\rho} |H(x,t,Z)|. \qquad (9\text{-}3)$$

In $U_C(x,t)$, the generalized solution is absolutely convergent, so that

$$|U(x,t)| \leq \sum_{n=0}^{\infty} |c_n| \; |p_n(x,t)| . \qquad (9\text{-}4)$$

Using (9-2) we get

$$|U(x,t)| \leq M_H(x,t,\rho) \sum_{n=0}^{\infty} |c_n|\rho^{-n} . \tag{9-5}$$

The series on the right converges for $C < \rho$, and we shall denote its sum by $\Delta_f(\rho)$. Therefore, for $C < \rho < R(x,t)$, the generalized solution has the bound

$$|U(x,t)| \leq M_H(x,t,\rho)\ \Delta_f(\rho) . \tag{9-6}$$

Now using the Cauchy-Schwarz inequality,

$$|c_n| = |(f,w_n)| \leq ||f||\ ||w_n|| , \tag{9-7}$$

where the norms are taken in the $L_2(I)$ sense. It follows that

$$\Delta_f(\rho) \leq ||f|| \sum_{n=0}^{\infty} ||w_n||\ \rho^{-n} = ||f||\Delta(\rho) . \tag{9-8}$$

The series in (9-8) converges for $\rho > \overline{\lim}|\,|w_n||^{1/n} = W$. So, if $W < R(x,t)$ in $I \times E$, we can pick ρ such that $\max(C,w) < \rho < R(x,t)$, and obtain the uniform bound

$$||U(x,t)| \leq ||f||\ [\Delta(\rho)M_H(x,t,\rho)] . \tag{9-9}$$

This bound gives a continuous dependence of the solution on initial data in the generalized sense.

In the case that (9-8) holds, and that $E = [0,\infty)$, these bounds may give stability results for the problems. This lemma follows from (9-6).

<u>Lemma 11-1</u>: If (9-8) holds, the null solution is stable if $M_H(x,t,\rho)$ is bounded as $t \longrightarrow \infty$, and is asymptotically stable if $M_H(x,t,\rho) \longrightarrow 0$ as $t \longrightarrow \infty$.

As an example, let

$$P(D_x,D_t) = D_x - D_t - 2t .$$

The formal solution operator is

$$F(t,D_x) = e^{-t^2}\ e^{tD_x} .$$

Then, for any Boas-Buck generator

$$H(x,t,a) = e^{-t^2}\, G(x+t,a).$$

It is clear that $M_H(x,t,\rho) \longrightarrow 0$ as $t \longrightarrow \infty$, unless $G(x,a)$ is entire in x with order ≥ 2. Therefore, the null solution is asymptotically stable if (9-8) holds.

10. The mixed initial and boundary value problem.

We wish to consider the mixed initial and boundary value problem in the strip $I \times E$, where $I = [a,b]$ and $E = [0,\infty)$. In order to properly pose this problem, with the equation of (2-1), we first impose Cauchy data on $I \times \{0\}$. Further we must specify K independent conditions on $\{a\} \times E$, and J conditions on $\{b\} \times E$. Here, K is the number of characteristics entering $I \times E$ at $(a,0)$, and J, the number of characteristics entering the strip at $(b,0)$ [5].

To illustrate the use of the above representation theory for this problem, assume that $K \geq 1$ and $J \geq 1$. Let us consider the problem,

$$P(D_x,D_t)U(x,t) = 0, \quad U(x,0) = f(x) \tag{10-1}$$

$$U(a,t) = G(t)\ , \qquad U(b,t) = H(t)\ .$$

It is clear that, if either K or J is greater than one, problem (10-1) is underdetermined.

Lemma 5-1 gives us an uncountably many representations for any functions $f(x) \in L_2(I)$ if the set $\{p_n(x)\}$ is minimal in $L_2(J)$ with $J \supset I$. Define $\varphi_g(x)$ by the following

$$\varphi_g(x) = \begin{cases} f(x)\ ; & x \in I \\ g(x)\ ; & x \in J - I\ . \end{cases} \tag{10-2}$$

The expansion of $\varphi_g(x)$ in terms of the set $\{p_n(x)\}$ is a representation of $f(x)$ for $x \varepsilon I$.

One might expect that, if $\Sigma a_n p_n(x)$ is a representation of zero in I, then $\Sigma a_n p_n(x,t)$ is a representation of zero in $I \times E$. A simple example shows that this is not the case. Let $F(t,D_x) = e^{tD_x}$, which is part of the wave equation operator. Then, for any $x \varepsilon I$, $p_n(x,t) = p_n(x+t)$. Let

$$\sum_{n=0}^{\infty} a_n p_n(x) = \begin{cases} 0 & ; x \varepsilon I \\ e^{-|x|} & ; x \not\varepsilon I. \end{cases}$$

Then, for $t > b - a$,

$$\sum_{n=0}^{\infty} a_n p_n(x+t) = e^{-|x+t|} \; ; x \varepsilon I .$$

This is obviously due to the propogation of solutions along characteristics for the hyperbolic problem. This gives us some freedom in adjusting the boundary values for the solutions.

Let $f(x)$ be the initial data function for the problem. Then,

$$\varphi_o(x) = \sum_{n=0}^{\infty} \varphi_{on} p_n(x)$$

is a representation for $f(x)$ in I which vanishes outside I. The generalized solution,

$$U_o(x,t) = \sum_{n=0}^{\infty} \varphi_{on} p_n(x,t) ,$$

will be called the basic solution to the problem. This solution imposes certain boundary values upon $\{a\} \times E$ and $\{b\} \times E$. Let

$$G_1(t) = G(t) - U_o(a,t) , \text{ and } H_1(t) = H(t) - U_o(b,t).$$

We must now solve the derived problem

$$\begin{aligned} &P(D_x,D_t)V(x,t) = 0, \quad V(x,0) = 0 \; ; x \varepsilon I, \\ &V(a,t) = G_1(t) , \qquad V(b,t) = H_1(t). \end{aligned} \tag{10-3}$$

We derive necessary and sufficient conditions on the functions $G_1(t)$

and $H_1(t)$ which allow us to solve (10-3) by means of the series representations derived above.

Assume that $V(x,t) = \Sigma\alpha_n p_n(x,t)$ is a generalized solution to (10-3). Then $V(a,t) = G_1(t)$, and it is necessary that $G_1(t) \in \overline{sp\{p_n(a,t)\}}$. Also, we need $H_1(t) \in \overline{sp\{p_n(b,t)\}}$. If $G_1(t)$ and $H_1(t)$ are in these manifolds, and if

$$G_1(t) = \Sigma\alpha_n p_n(a,t) \; ; \quad H_1(t) = \Sigma\beta_n p_n(b,t) \; ,$$

the sequence $\{\alpha_n - \beta_n\}$ must determine a zero representation in either $\overline{sp\,\{p_n(a,t)\}}$ or $\overline{sp\,\{p_n(b,t)\}}$. If this last condition is satisfied, let $\{\gamma_n\}$ denote the common sequence of coefficients. Then, $V(x,0) = \Sigma\gamma_n p_n(x)$ must vanish for $x \in I$. It is clear that these steps are reversible, so we have the desired conditions.

<u>Lemma 10-1</u>: Problem (10-3) has a solution in the generalized sense of the representation theory if and only if the following three conditions are simultaneously satisfied.

$$G_1(t) \in \overline{sp\{p_n(a,t)\}} \; , \quad H_1(t) \in \overline{sp\{p_n(b,t)\}} \; . \tag{10-4}$$

There exists $\{\gamma_n\}$ such that

$$\text{(a)} \quad G_1(t) = \sum_{n=0}^{\infty} \gamma_n p_n(a,t), \quad \text{and} \tag{10-5}$$

$$\text{(b)} \quad H_1(t) = \sum_{n=0}^{\infty} \gamma_n p_n(b,t) \; , \quad \text{and}$$

$$\sum_{n=0}^{\infty} \gamma_n p_n(x) = 0 \quad \text{for} \quad x \in I \; . \tag{10-6}$$

It is clear that (10-3) may have a solution which is not attainable from this theory, or that (10-3) may not have a solution even though (10-1) does, since the theory may not be applicable to the particular problem.

For the heat equation, with $G(x,a) = e^{ax}$, we get

$$p_n(x,t) = \sum_{m=0}^{[n/2]} \frac{t^m x^{n-2m}}{m!\,(n-2m)!} \quad .$$

Therefore, both $p_{2n}(x,t)$ and $p_{2n+1}(x,t)$ have degree n in t. Therefore, $sp\{p_n(a,t)\}$ and $sp\{p_n(b,t)\}$ are dense on $L_2(E)$. Therefore, condition (10-4) is simply that $g_1(t)$ and $H_1(t)$ are elements of $L_2(E)$, in this case. Also, by the duplication of degree, $\{p_n(a,t)\}$ and $\{p_n(b,t)\}$ cannot be minimal, so there are zero representations available to facilitate (10-5).

11. The nonhomogeneous problem.

We now indicate the analogous operator-expansion theory for the problem

$$P(D_x,D_t)U(x,t) = f(x,t) \tag{11-1}$$

$$U(x,0) = g(x). \tag{11-2}$$

In case $P(D_x,D_t)$ is of first order in D_t, we may use Duhamel's principle [5] to solve this problem. However, an attempt at extending Duhamel's principle to the higher order case yields integro-differential equations which may not be solvable. However, suppose there exists a formal differential operator $G(D_x,D_t)$ such that

$$P(D_xD_t)G(D_x,D_t)\ f(x,t) = f(x,t). \tag{11-3}$$

then, $G(\lambda,\mu)$ is the solution to the associated equation

$$P(D_x + \lambda,D_t + \mu)G(\lambda,\mu) = 1. \tag{11-4}$$

Further, just as in section 2, if there is a solution to the equation

$$P(D_x,D_t)V_{mn}(x,t) = x^m t^n \tag{11-5}$$

for each pair (m,n), then the formal operator exists in the form

$$G(D_x,D_t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} G_{mn}(x,t)D_x^m D_t^t , \tag{11-6}$$

and $V_{mn}(x,t) = G(D_x,D_t)x^m t^n$. Therefore, we have the theorem.

<u>Theorem 11-1</u>: If (11-5) has a solution for each m and n, then there exists a formal operator, $G(D_x,D_t)$, such that (11-3) holds for any polynomial $f(x,t)$.

We may now expand arbitrary L_2 functions $f(x,t)$, just as in the homogeneous case, in terms of polynomials of the form $p_m(x)q_n(t)$, and get formal solutions of the form

$$\hat{u}(x,t) \backsim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{mn}P_{mn}(x,t) , \tag{11-7}$$

where now $P_{mn}(x,t) = G(D_x,D_t)p_m(x)q_n(t)$. Then, let $h(x) = g(x) - \hat{U}(x,0)$. Let $F(D_x)$ be the formal solution operator for the homogeneous problem. That gives

$$P(D_x,D_t)F(D_x)h(x) = 0 \quad \text{with}$$
$$F(D_x)h(x) \Big|_{t=0} = h(x) . \tag{11-8}$$

Then, the formal solution to problem (11-1,2) may be written as

$$U(x,t) = G(D_x,D_t)f(x,t) + F(D_x)h(x). \tag{11-9}$$

Convergence of the formal series involved is to be handled just as in section 6.

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